

# FELL BUNDLES AND IMPRIMITIVITY THEOREMS: TOWARDS A UNIVERSAL GENERALIZED FIXED POINT ALGEBRA

S. KALISZEWSKI, PAUL S. MUHLY, JOHN QUIGG, AND DANA P. WILLIAMS

**ABSTRACT.** We apply the One-Sided Action Theorem from the first paper in this series to prove that Rieffel's Morita equivalence between the reduced crossed product by a proper saturated action and the generalized fixed-point algebra is a quotient of a Morita equivalence between the full crossed product and a “universal” fixed-point algebra. We give several applications, to Fell bundles over groups, reduced crossed products as fixed-point algebras, and  $C^*$ -bundles.

## 1. INTRODUCTION

The One-Sided Action Theorem [4, Corollary 2.3] goes as follows: let  $\mathcal{A} \rightarrow \mathcal{X}$  be a Fell bundle over a locally compact groupoid, and let  $G$  be a locally compact group. If  $G$  acts freely and properly on  $\mathcal{A}$ , then the Banach bundle  $\mathcal{A} \rightarrow \mathcal{X}$  gives a Yamagami equivalence between the semidirect-product Fell bundle  $\mathcal{A} \rtimes G \rightarrow \mathcal{X} \rtimes G$  and the orbit Fell bundle  $G \backslash \mathcal{A} \rightarrow G \backslash \mathcal{X}$ . In the current paper we will connect this quotient equivalence with Rieffel's imprimitivity theorem for generalized-fixed-point-algebras [9, Corollary 1.7]. By [7, Theorem 6.4],  $\Gamma_c(\mathcal{A})$  completes to give a  $C^*(\mathcal{A} \rtimes G) - C^*(G \backslash \mathcal{A})$  imprimitivity bimodule  $X$ . By [5, Theorem 7.1] there is an associated action  $\alpha : G \rightarrow \text{Aut } C^*(\mathcal{A})$ , and an isomorphism

$$C^*(\mathcal{A} \rtimes G) \cong C^*(\mathcal{A}) \rtimes_{\alpha} G,$$

so  $X$  may be viewed as a  $C^*(\mathcal{A}) \rtimes_{\alpha} G - C^*(G \backslash \mathcal{A})$  imprimitivity bimodule.

We will show that the action  $\alpha$  is proper and saturated in Rieffel's sense, so that Rieffel's theorem gives a  $C^*(\mathcal{A}) \rtimes_{\alpha, r} G - C^*(\mathcal{A})^{\alpha}$  imprimitivity bimodule  $X_R$ . Since  $C^*(\mathcal{A}) \rtimes_{\alpha, r} G$  is a quotient of  $C^*(\mathcal{A}) \rtimes_{\alpha} G$ , it seems natural to guess that the imprimitivity bimodule  $X_R$  is a quotient of  $X$ , and we will verify this in Theorem 3.1 below.

Thus, in some sense  $C^*(G \backslash \mathcal{A})$  can be regarded as a “universal”, or “full” version of a generalized fixed-point algebra, whereas Rieffel's fixed-point algebra is in some sense a “reduced” version. More precisely, Rieffel's generalized fixed-point algebra is Morita equivalent to a *reduced* crossed product, while our “universal” fixed-point algebra is Morita equivalent to the associated full crossed product.

We begin in Section 2 with some preliminaries on transformation Fell bundles.

Section 3 contains our main result on the Rieffel Surjection, and we further show that the quotient map of our “universal fixed-point algebra”  $C^*(G \backslash \mathcal{A})$  onto the “reduced one”  $C^*(\mathcal{A})^{\alpha}$  can be identified with the regular representation of  $C^*(G \backslash \mathcal{A})$ .

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In Section 4 we give three applications: (1) for a Fell bundle  $\mathcal{B}$  over a locally compact group  $G$ , we show that (as has surely been suspected by the cognoscenti) the regular representation of  $C^*(\mathcal{B})$  is a normalization of the canonical coaction, (2) the reduced crossed product by a  $C^*$ -action can be viewed as a generalized fixed-point algebra, and (3) for a  $C^*$ -bundle over a space, the full and reduced fixed-point algebras coincide.

## 2. PRELIMINARIES

We adopt the conventions of [5, 4]. All our Banach bundles will be upper semi-continuous and separable, all our spaces and groupoids will be locally compact Hausdorff and second countable, and our groupoids will all have left Haar systems.

In [5, Proposition 1.7], building upon a result in [7], we proved that if  $\mathcal{A} \rightarrow \mathcal{X}$  is a Fell bundle over a groupoid and  $X_0$  is a dense subspace of a Hilbert  $C^*$ -module  $X$ , and if we are given linear maps  $\{L_0(f) : f \in \Gamma_c(\mathcal{A})\}$  on  $X_0$  such that

- (i)  $L_0(f)L_0(g) = L_0(fg)$  for all  $f, g \in \Gamma_c(\mathcal{A})$ ,
- (ii)  $\langle L_0(f)x, y \rangle = \langle x, L_0(f^*)y \rangle$  for all  $f \in \Gamma_c(\mathcal{A})$ ,  $x, y \in X_0$ ,
- (iii)  $f \mapsto \langle L_0(f)x, y \rangle$  is inductive-limit continuous for all  $x, y \in X_0$ , and
- (iv)  $\text{span}\{L_0(f)x : f \in \Gamma_c(\mathcal{A}), x \in X_0\}$  is dense in  $X$ ,

then  $L_0$  extends uniquely to a nondegenerate homomorphism  $L : C^*(\mathcal{A}) \rightarrow \mathcal{L}(X)$ . In some applications of this result we will have  $X = C^*(\mathcal{B})$  and  $X_0 = \Gamma_c(\mathcal{B})$  for some Fell bundle  $\mathcal{B}$  (where  $C^*(\mathcal{B})$  is regarded as a Hilbert module over itself in the canonical way). Another special case which can arise is where  $\mathcal{X}$  is a trivial groupoid and  $\mathcal{A}$  is a single  $C^*$ -algebra  $A$ , in which case (iii) means that  $a \mapsto \langle L(a)x, y \rangle$  is norm continuous for all  $x, y \in X_0$ .

Given a Fell bundle  $\mathcal{B} \rightarrow \mathcal{Y}$  over a groupoid, and an action of  $\mathcal{Y}$  on a space  $\Omega$ , in [4, Section A.1] we defined a transformation Fell bundle  $\mathcal{B} * \Omega \rightarrow \mathcal{Y} * \Omega$ . We will need to know a little more about this construction here:

**Proposition 2.1.** *Let  $\mathcal{B} \rightarrow \mathcal{Y}$  be a Fell bundle over a groupoid, and let  $\mathcal{Y}$  act on a space  $\Omega$ . Then there are nondegenerate homomorphisms  $\Phi : C^*(\mathcal{B}) \rightarrow M(C^*(\mathcal{B} * \Omega))$  and  $\mu : C_0(\Omega) \rightarrow M(C^*(\mathcal{B} * \Omega))$  such that if  $f \in \Gamma_c(\mathcal{B})$ ,  $\phi \in C_0(\Omega)$ , and  $a \in \Gamma_c(\mathcal{B} * \Omega)$ , then  $\Phi(f)a$  and  $\mu(\phi)a$  are the sections in  $\Gamma_c(\mathcal{B} * \Omega)$  determined by*

$$(2.1) \quad (\Phi(f)a)_1(y, u) = \int_{\mathcal{Y}} f(x)a_1(x^{-1}y, u) d\lambda^{r(y)}(x).$$

$$(2.2) \quad (\mu(\phi)a)_1(y, u) = \phi(y \cdot u)a_1(y, u).$$

Moreover,

$$(2.3) \quad \Phi(f)\mu(\phi) = f \boxtimes \phi,$$

where  $f \boxtimes \phi$  is the section in  $\Gamma_c(\mathcal{B} * \Omega)$  determined by

$$(f \boxtimes \phi)_1(y, u) = f(y)\phi(u),$$

and we have

$$(2.4) \quad \overline{\text{span}}\{\Phi(C^*(\mathcal{B}))\mu(C_0(\Omega))\} = C^*(\mathcal{B} * \Omega).$$

*Proof.* To establish the existence of the nondegenerate homomorphisms  $\Phi$  and  $\mu$ , we will apply the above-mentioned extension result [5, Proposition 1.7] with  $X = C^*(\mathcal{B} * \Omega)$  and  $X_0 = \Gamma_c(\mathcal{B} * \Omega)$ .

We begin with  $\Phi$ . For each  $f \in \Gamma_c(\mathcal{B})$ , (2.1) defines a linear map  $\Phi_0(f)$  on  $\Gamma_c(\mathcal{B} * \Omega)$ . If we can show that

- (i)  $\Phi_0$  is multiplicative,
- (ii)  $\langle \Phi_0(f)a, b \rangle = \langle a, \Phi_0(f^*)b \rangle$ ,
- (iii)  $f \mapsto \langle \Phi_0(f)a, b \rangle$  is inductive-limit continuous, and
- (iv)  $\text{span}\{\Phi_0(f)a : f \in \Gamma_c(\mathcal{B}), a \in \Gamma_c(\mathcal{B} * \Omega)\}$  is inductive-limit dense in  $\Gamma_c(\mathcal{B} * \Omega)$ ,

then it will follow that  $\Phi_0$  extends uniquely to a nondegenerate homomorphism  $\Phi : C^*(\mathcal{B}) \rightarrow M(C^*(\mathcal{B} * \Omega))$ .

For (i), if  $f, g \in \Gamma_c(\mathcal{B})$  and  $a \in \Gamma_c(\mathcal{B} * \Omega)$  we have

$$\begin{aligned}
 & (\Phi_0(f)\Phi_0(g)a)_1(y, u) \\
 &= \int_{\mathcal{Y}} f(x)(\Phi_0(g)a)_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} f(x) \int_{\mathcal{Y}} g(z)a_1(z^{-1}x^{-1}y, u) d\lambda^{r(x^{-1}y)}(z) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} f(x)g(x^{-1}z)a_1(z^{-1}y, u) d\lambda^{r(y)}(z) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} f(x)g(x^{-1}z) d\lambda^{r(y)}(x) a_1(z^{-1}y, u) d\lambda^{r(y)}(z) \\
 &= \int_{\mathcal{Y}} (f * g)(z)a_1(z^{-1}y, u) d\lambda^{r(y)}(z) \\
 &= (\Phi_0(f * g)a)_1(y, u).
 \end{aligned}$$

For (ii), if  $f \in \Gamma_c(\mathcal{B})$  and  $a, b \in \Gamma_c(\mathcal{B} * \Omega)$  we have

$$\begin{aligned}
 & (\langle \Phi_0(f)a, b \rangle_{\Gamma_c(\mathcal{B} * \Omega)})_1(y, u) \\
 &= ((\Phi_0(f)a)^* * b)_1(y, u) \\
 &= \int_{\mathcal{Y}} (\Phi_0(f)a)_1^*(x, x^{-1}y \cdot u) b_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} (\Phi_0(f)a)_1(x^{-1}, y \cdot u)^* b_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \left( \int_{\mathcal{Y}} f(z)a_1(z^{-1}x^{-1}, y \cdot u) d\lambda^{r(x^{-1})}(z) \right)^* b_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}x^{-1}, y \cdot u)^* f(z)^* d\lambda^{s(x)}(z) b_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* f(x^{-1}z)^* b_1(x^{-1}y, u) d\lambda^{r(x)}(z) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* f(x^{-1}z)^* b_1(x^{-1}y, u) d\lambda^{r(y)}(z) d\lambda^{r(y)}(x) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* f(x^{-1}z)^* b_1(x^{-1}y, u) d\lambda^{r(y)}(x) d\lambda^{r(y)}(z) \\
 &= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* f(x^{-1}z)^* b_1(x^{-1}y, u) d\lambda^{r(z)}(x) d\lambda^{r(y)}(z)
 \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* f(x^{-1})^* b_1(x^{-1}z^{-1}y, u) d\lambda^{s(z)}(x) d\lambda^{r(y)}(z) \\
&= \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* \int_{\mathcal{Y}} f^*(x) b_1(x^{-1}z^{-1}y, u) d\lambda^{r(z^{-1}y)}(x) d\lambda^{r(y)}(z) \\
&= \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* (\Phi_0(f^*)b)_1(z^{-1}y, u) d\lambda^{r(y)}(z) \\
&= (a^* * (\Phi_0(f^*)b))_1(y, u) \\
&= (\langle a, \Phi_0(f^*)b \rangle_{\Gamma_c(\mathcal{B} * \Omega)})_1(y, u).
\end{aligned}$$

For (iii), let  $K \subset \mathcal{Y}$  be compact, and let

$$\Gamma_K(\mathcal{B}) = \{f \in \Gamma_c(\mathcal{B}) : \text{supp } f \subset K\}.$$

For  $f \in \Gamma_K(\mathcal{B})$  and  $a, b \in \Gamma_c(\mathcal{B} * \Omega)$ , the inner product  $\langle \Phi_0(f)a, b \rangle$  in this part is to be interpreted in  $C^*(\mathcal{B} * \Omega)$ . It suffices to show that for fixed  $a, b \in \Gamma_c(\mathcal{B} * \Omega)$  the linear map

$$f \mapsto \langle \Phi_0(f)a, b \rangle : \Gamma_K(\mathcal{B}) \rightarrow C^*(\mathcal{B} * \Omega)$$

is bounded when  $\Gamma_K(\mathcal{B})$  is given the uniform norm  $\|\cdot\|_u$ , and for this it suffices to show that this linear map actually takes values in  $\Gamma_P(\mathcal{B} * \Omega)$  for some compact set  $P \subset \mathcal{Y} * \Omega$  and is bounded when  $\Gamma_P(\mathcal{B} * \Omega)$  is given its uniform norm  $\|\cdot\|_u$ .

Choose compact sets  $L \subset \mathcal{Y}$  and  $M \subset \mathcal{Y}^0 * \Omega$  such that both  $a$  and  $b$  are supported in  $L \times M$ . The computation in (ii) shows that for  $a, b \in \Gamma_c(\mathcal{B} * \Omega)$  we have

$$\begin{aligned}
&(\langle \Phi_0(f)a, b \rangle_{\Gamma_c(\mathcal{B} * \Omega)})_1(y, u) \\
&= \int_{\mathcal{Y}} \int_{\mathcal{Y}} a_1(z^{-1}, y \cdot u)^* f(x^{-1}z)^* b_1(x^{-1}y, u) d\lambda^{r(y)}(z) d\lambda^{r(y)}(x),
\end{aligned}$$

so for the left-hand side to be nonzero we must have  $u \in M$ . Then for the integration we can assume that  $z^{-1} \in L$  and  $x^{-1}z \in K$ , so  $x \in L^{-1}K^{-1}$ , and that  $x^{-1}y \in L$ , so  $y \in L^{-1}K^{-1}L$ . Thus  $\langle \Phi_0(f)a, b \rangle_{\Gamma_c(\mathcal{B} * \Omega)}$  is supported in the compact set

$$P := (L^{-1}K^{-1}L) * M \subset \mathcal{Y} * \Omega.$$

We have

$$\begin{aligned}
&\left\| (\langle \Phi_0(f)a, b \rangle_{\Gamma_c(\mathcal{B} * \Omega)})_1(y, u) \right\| \\
&\leq \int_{L^{-1}K^{-1}} \int_{L^{-1}} \|a_1(z^{-1}, y \cdot u)\| \|f(x^{-1}z)\| \|b_1(x^{-1}y, u)\| \\
&\quad d\lambda^{r(y)}(z) d\lambda^{r(y)}(x) \\
&\leq \|a\|_u \|f\|_u \|b\|_u \lambda^{r(y)}(L^{-1}K^{-1}) \lambda^{r(y)}(L^{-1}).
\end{aligned}$$

Since the sets  $L^{-1}K^{-1}$ ,  $L^{-1}$ , and  $L^{-1}K^{-1}L$  are compact, by the properties of Haar systems there is a constant  $c$  such that

$$\lambda^{r(y)}(L^{-1}K^{-1}) \lambda^{r(y)}(L^{-1}) \leq c \quad \text{for all } y \in L^{-1}K^{-1}L,$$

giving the required boundedness.

For (iv), first note that the properties of Banach bundles imply that

$$(2.5) \quad \Gamma_c(\mathcal{B}) \boxtimes C_0(\Omega) \text{ is inductive-limit dense in } \Gamma_c(\mathcal{B} * \Omega).$$

Moreover, it is clear that if  $f, g \in \Gamma_c(\mathcal{B})$  and  $\phi \in C_c(\mathcal{Y}^0 * \Omega)$  then

$$\Phi_0(f)(g \otimes \phi) = (f * g) \boxtimes \phi.$$

Since  $\text{span}\{f * g : f, g \in \Gamma_c(\mathcal{B})\}$  is inductive-limit dense in  $\Gamma_c(\mathcal{B})$ , and since for fixed  $\phi \in C_c(\mathcal{Y}^0 * \Omega)$  the linear map

$$g \mapsto g \boxtimes \phi : \Gamma_c(\mathcal{B}) \rightarrow \Gamma_c(\mathcal{B} * \Omega)$$

is inductive-limit continuous, the required density follows.

We have proved the existence of  $\Phi$  satisfying (2.1). The proof of the existence of  $\mu$  satisfying (2.2) is similar, but easier. Again, for each  $\phi \in C_0(\Omega)$  (2.2) defines a linear map  $\mu_0(\phi)$  on  $\Gamma_c(\mathcal{B} * \Omega)$ , and we must verify appropriate versions of the properties (i)–(iv). Of these, (i) and (iii) are obvious, and (iv) follows from density of  $\Gamma_c(\mathcal{B}) \boxtimes C_c(\Omega)$ . We give the computation for (ii):

$$\begin{aligned} \langle \mu_0(f)a, b \rangle_1(y, u) &= \int_{\mathcal{Y}} (\mu_0(f)a)_1^*(x, x^{-1}y \cdot u) b_1(x^{-1}y, u) \lambda^{r(y)}(x) \\ &= \int_{\mathcal{Y}} (\mu_0(f)a)_1(x^{-1}, y \cdot u)^* b_1(x^{-1}y, u) \lambda^{r(y)}(x) \\ &= \int_{\mathcal{Y}} a_1(x^{-1}, y \cdot u)^* \overline{f(x^{-1}y \cdot u)} b_1(x^{-1}y, u) \lambda^{r(y)}(x) \\ &= \int_{\mathcal{Y}} a_1^*(x, x^{-1}y \cdot u) (\mu_0(\bar{f})b)_1(x^{-1}y, u) \lambda^{r(y)}(x) \\ &= \langle a, \mu_0(\bar{f})b \rangle(y, u). \end{aligned}$$

Finally, (2.3) is a simple computation, and then (2.4) follows from (2.5).  $\square$

### 3. THE RIEFFEL SURJECTION

Our main result is the following application of the One-Sided Action theorem:

**Theorem 3.1** (Rieffel Surjection). *Let  $p : \mathcal{A} \rightarrow \mathcal{X}$  be a Fell bundle over a locally compact groupoid, and let  $G$  be a locally compact group. Suppose that  $G$  acts freely and properly on (the left of)  $\mathcal{A}$  by automorphisms, so that we also have an associated action  $\alpha : G \rightarrow \text{Aut } C^*(\mathcal{A})$ . Then there exist maps  $\Upsilon$  and  $\Phi$  such that*

$$(\Lambda, \Upsilon, \Phi) : (C^*(\mathcal{A}) \rtimes_{\alpha} G, X, C^*(G \backslash \mathcal{A})) \rightarrow (C^*(\mathcal{A}) \rtimes_{\alpha, r} G, X_R, C^*(\mathcal{A})^{\alpha})$$

*is a surjection of imprimitivity bimodules, where  $\Lambda$  is the regular representation.*

The above theorem will be proven in the following equivalent form, rephrased using the principal-bundle decomposition [4, Theorem A.11]:

**Theorem 3.2** (Rieffel Surjection). *Let  $p : \mathcal{B} \rightarrow \mathcal{Y}$  be a Fell bundle over a locally compact groupoid, and let  $G$  be a locally compact group. Suppose that both  $\mathcal{Y}$  and  $G$  act on (the left of) a locally compact Hausdorff space  $\Omega$ , and that the action of  $G$  is free and proper and commutes with the  $\mathcal{Y}$ -action, so that  $G$  also acts freely and properly by automorphisms on the transformation Fell bundle  $\mathcal{B} * \Omega \rightarrow \mathcal{Y} * \Omega$ , and we also have an associated action  $\alpha : G \rightarrow \text{Aut } C^*(\mathcal{B} * \Omega)$ . Then there exist maps  $\Upsilon$  and  $\Phi$  such that*

$$(\Lambda, \Upsilon, \Phi) : (C^*(\mathcal{B} * \Omega) \rtimes_{\alpha} G, X, C^*(\mathcal{B})) \rightarrow (C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G, X_R, C^*(\mathcal{B} * \Omega)^{\alpha})$$

*is a surjection of imprimitivity bimodules, where  $\Lambda$  is the regular representation.*

*Remark.* Obviously Theorem 3.1 implies Theorem 3.2; to see that the two results are in fact equivalent, just use the principal-bundle decomposition

$$\mathcal{A} \cong \mathcal{B} * \Omega$$

from [4, Theorem A.11].

Our strategy for proving Theorem 3.2 will be to take  $\Upsilon$  as a suitable extension of the identity map on  $\Gamma_c(\mathcal{B} * \Omega)$ . This makes sense, since both imprimitivity bimodules  $X$  and  $X_R$  are completions of  $\Gamma_c(\mathcal{B} * \Omega)$ , in the latter case because the action  $\alpha$  of  $G$  on  $C^*(\mathcal{B} * \Omega)$  is saturated and proper with respect to the dense \*-subalgebra  $\Gamma_c(\mathcal{B} * \Omega)$ .

We first verify that the homomorphism  $\Phi$  from Proposition 2.1 is the one we want:

**Proposition 3.3.** *With the hypotheses of Theorem 3.2, the homomorphism  $\Phi : C^*(\mathcal{B}) \rightarrow M(C^*(\mathcal{B} * \Omega))$  of Proposition 2.1 maps onto the generalized fixed-point algebra  $C^*(\mathcal{B} * \Omega)^\alpha$ .*

*Proof.* The proof will be rather long, and we break it into steps.

*Step 1.* We first need to know that the generalized fixed-point algebra exists, and for the proof of Theorem 3.2 we further want to know that this fixed-point algebra is Morita equivalent to the reduced crossed product; by [9, Corollary 1.7], we can accomplish this by showing that the action  $\alpha$  of  $G$  on  $C^*(\mathcal{B} * \Omega)$  is proper and saturated with respect to the dense \*-subalgebra  $\Gamma_c(\mathcal{B} * \Omega)$ . Recall that there is a nondegenerate embedding of  $C_0(\mathcal{X}^0)$  in  $M(C^*(\mathcal{B} * \Omega))$  determined by

$$(\phi \cdot a)(x) = \phi(r(x))a(x) \quad \text{and} \quad (a \cdot \phi)(x) = a(x)\phi(s(x))$$

for  $\phi \in C_0(\mathcal{X}^0)$  and  $a \in \Gamma_c(\mathcal{B} * \Omega)$ . Moreover, this embedding is  $G$ -equivariant. Therefore properness and saturatedness follow from [10, Theorem 5.7] and [3, Lemma 4.1].

*Step 2.* For each  $a \in \Gamma_c(\mathcal{B} * \Omega)$ , we will show that there exists a unique section  $\Psi(a) \in \Gamma_c(\mathcal{B})$  such that

$$\Psi(a)(y) = \int_G a_1(y, s^{-1} \cdot u) ds,$$

where  $u \in \Omega$  is any element satisfying  $q(u) = s(y)$ .

It is clear that the value of the integral is a well-defined element of  $B(y)$ , because for any  $u \in q^{-1}(s(y))$  the map  $s \mapsto a_1(y, s^{-1} \cdot u)$  is in  $C_c(G, B(y))$ , and by left-invariance of the Haar measure on  $G$  the value of the integral is independent of the choice of  $u$ . It is also clear that the integral is zero for  $y$  outside the compact subset  $q(\text{supp } a)$  of  $\mathcal{Y}$ . It remains to see that  $\Psi(a)$  is continuous. For this purpose we first show continuity of the auxiliary function  $\tilde{\Psi}(a) : \mathcal{X} \rightarrow \mathcal{B} * \Omega$  defined by

$$\tilde{\Psi}(a)(y, u) = \int_G (a_1(y, s^{-1} \cdot u), u) ds.$$

Once we have shown this, it will be clear that in fact  $\tilde{\Psi}(a)$  is a section of the bundle  $p : \mathcal{B} * \Omega \rightarrow \mathcal{X}$ , and that

$$\tilde{\Psi}(a)_1(y, u) = \int_G a_1(y, s^{-1} \cdot u) ds.$$

Since the function  $\tilde{\Psi}(a)_1$  is obviously independent of the second variable, we will be able to conclude that

$$\Psi(a) \circ q = \tilde{\Psi}(a),$$

and hence that the function  $\Psi(a)$  is continuous as well, because  $q : \mathcal{X} \rightarrow \mathcal{Y}$  is a quotient map.

Fix  $(y_0, u_0) \in \mathcal{X}$ , and choose a compact neighborhood  $U$  of  $(y_0, u_0)$  and then a function  $g \in C_c(\mathcal{X})$  that is identically 1 on  $U$ . Then  $\tilde{\Psi}(a) = \tilde{\Psi}(a)g$  on  $U$ , so to show that  $\tilde{\Psi}(a)$  is continuous at  $(y_0, u_0)$  it suffices to show that  $\tilde{\Psi}(a)g$  is continuous. Let  $K = \text{supp } g$ , and define

$$\Gamma_K(\mathcal{B} * \Omega) = \{c \in \Gamma_c(\mathcal{B} * \Omega) : \text{supp } c \subset K\},$$

which is a Banach space with the sup norm. Now define  $\psi \in C_c(G, \Gamma_K(\mathcal{B} * \Omega))$  by

$$\psi(s)(y, u) = a(y, s^{-1}u)g(y, u).$$

Then the integral  $\int_G \psi(s) ds$  is norm-convergent in the Banach space  $\Gamma_K(\mathcal{B} * \Omega)$ , and it is routine to check that it agrees with  $\tilde{\Psi}(a)g$ , and this completes Step 2.

*Step 3.* We recall that

$$C^*(\mathcal{B} * \Omega)^\alpha = \overline{E(\Gamma_c(\mathcal{B} * \Omega))},$$

where  $E$  is the conditional expectation from [6], defined as follows: for  $a \in \Gamma_c(\mathcal{B} * \Omega)$ ,  $E(a)$  is the unique element of  $M(C^*(\mathcal{B} * \Omega))$  such that for all  $\omega \in C^*(\mathcal{B} * \Omega)^*$  we have

$$\omega(E(a)) = \int_G \omega(\alpha_s(a)) ds.$$

We will show that if  $a, b \in \Gamma_c(\mathcal{B} * \Omega)$  then  $E(a)b$  coincides with the section in  $\Gamma_c(\mathcal{B} * \Omega)$  given by  $\Phi \circ \Psi(a)b$ .

We first show that  $E(a)b$  is an integral in  $C^*(\mathcal{B} * \Omega)$ , more precisely

$$(3.1) \quad E(a)b = \int_G^{C^*(\mathcal{B} * \Omega)} \alpha_s(a)b ds,$$

where the superscript “ $C^*(\mathcal{B} * \Omega)$ ” on the integral sign indicates that this is a norm-convergent integral in  $C^*(\mathcal{B} * \Omega)$ . To verify (3.1), let  $\omega \in C^*(\mathcal{B} * \Omega)^*$ , and define  $b \cdot \omega \in C^*(\mathcal{B} * \Omega)^*$  by

$$b \cdot \omega(a) = \omega(ab).$$

Then

$$\begin{aligned} \omega(E(a)b) &= b \cdot \omega(E(a)) \\ &= \int_G b \cdot \omega(\alpha_s(a)) ds \\ &= \int_G \omega(\alpha_s(a)b) ds \\ &= \omega \left( \int_G^{C^*(\mathcal{B} * \Omega)} \alpha_s(a)b ds \right), \end{aligned}$$

because  $s \mapsto \alpha_s(a)b$  is in  $C_c(G, C^*(\mathcal{B} * \Omega))$ .

Our strategy is to identify the integral in (3.1) with one in  $\Gamma_c(\mathcal{B} * \Omega)$ . Define  $g : G \rightarrow \Gamma_c(\mathcal{B} * \Omega)$  by

$$g(s) = \alpha_s(a)b.$$

Then there exist compact sets  $C \subset G$  and  $K \subset \mathcal{X}$  such that

$$g(s)(x) = 0 \quad \text{if } (s, x) \notin C \times K.$$

We can view

$$g \in C_c(G, \Gamma_K(\mathcal{B} * \Omega)),$$

and hence we can integrate this map, getting an element

$$c := \int_G^{\Gamma_K(\mathcal{B} * \Omega)} g(s) ds$$

of  $\Gamma_K(\mathcal{B} * \Omega)$ . Let  $j : \Gamma_K(\mathcal{B} * \Omega) \rightarrow A$  be the inclusion map. Then  $j$  is bounded, and we have

$$\begin{aligned} j(c) &= j \left( \int_G^{\Gamma_K(\mathcal{B} * \Omega)} g(s) ds \right) \\ &= \int_G^A j(g(s)) ds \\ &= E(a)b, \end{aligned}$$

showing that  $E(a)b$  is the section in  $\Gamma_c(\mathcal{B} * \Omega)$  given by

$$(E(a)b)(y, u) = \int_G (\alpha_s(a)b)(y, u) ds.$$

To show that  $E(a)b = \Phi \circ \Psi(a)b$ , note that since evaluation at  $(y, u) \in \mathcal{X}$  is a continuous linear map of  $\Gamma_K(\mathcal{B} * \Omega)$  into the fibre  $B(y) \times \{u\}$ , it follows that

$$\begin{aligned} (E(a)b)_1(y, u) &= \int_G (\alpha_s(a)b)_1(y, u) ds \\ &= \int_G \int_{\mathcal{Y}} \alpha_s(a)_1(x, x^{-1}y \cdot u) b_1(x^{-1}y, u) d\lambda^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} a_1(x, s^{-1} \cdot x^{-1}y \cdot u) b_1(x^{-1}y, u) d\lambda^{r(y)}(x) ds \\ &= \int_{\mathcal{Y}} \int_G a_1(x, s^{-1} \cdot x^{-1}y \cdot u) b_1(x^{-1}y, u) ds d\lambda^{r(y)}(x) \\ &= \int_{\mathcal{Y}} \int_G a_1(x, s^{-1} \cdot x^{-1}y \cdot u) ds b_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\ &= \int_{\mathcal{Y}} \Psi(a)(x) b_1(x^{-1}y, u) d\lambda^{r(y)}(x) \\ &= (\Phi(\Psi(a))b)_1(y, u). \end{aligned}$$

This completes Step 3.

*Step 4.* We show that  $\Psi(\Gamma_c(\mathcal{B} * \Omega))$  is inductive-limit dense in  $\Gamma_c(\mathcal{B})$ .

Clearly  $\Psi(\Gamma_c(\mathcal{B} * \Omega))$  is a  $C_0(\mathcal{Y})$ -module, so it suffices to show that its fibres are full, i.e., for  $y \in \mathcal{Y}$  and  $c \in B(y)$  we can find a section in  $\Psi(\Gamma_c(\mathcal{B} * \Omega))$  whose value at  $y$  is  $c$ . Pick  $b \in \Gamma_c(\mathcal{B})$  such that  $b(y) = c$ . Next choose  $u \in \Omega$  such that  $q(u) = s(y)$ , then choose a nonnegative function  $g \in C_c(\Omega)$  such that

$$\int_G g(s^{-1} \cdot u) ds = 1.$$

To see that such a  $g$  exists, note that  $s \mapsto s^{-1} \cdot u$  is a homeomorphism of  $G$  onto the closed subset  $G \cdot u$  of  $\Omega$ , and we can choose a nonnegative function  $g_0 \in C_c(G \cdot u)$



such that  $\int_G g_0(s^{-1} \cdot x_0) ds = 1$ , and then use Tietze's theorem to extend  $g_0$  to  $g \in C_c(\Omega)$ . Let  $b \boxtimes g$  denote the element of  $\Gamma_c(\mathcal{B} * \Omega)$  defined by

$$(b \boxtimes g)(y, s) = (b(y)g(s), s).$$

Then

$$\begin{aligned} \Psi(b \boxtimes g)(y) &= \int_G (b \boxtimes g)_1(y, s^{-1} \cdot u) ds \\ &= \int_G b(y)g(s^{-1} \cdot u) ds \\ &= b(y) \int_G g(s^{-1} \cdot u) ds \\ &= b(y) \\ &= c. \end{aligned}$$

*Step 5.* Proposition 3.3 now follows quickly from the above: we have a homomorphism  $\Phi : C^*(\mathcal{B}) \rightarrow M(C^*(\mathcal{B} * \Omega))$  such that

$$\Phi(\Psi(\Gamma_c(\mathcal{B} * \Omega))) = E(\Gamma_c(\mathcal{B} * \Omega)).$$

Since  $\Psi(\Gamma_c(\mathcal{B} * \Omega))$  is dense in  $C^*(\mathcal{B})$  via the composition of inclusions

$$\Psi(\Gamma_c(\mathcal{B} * \Omega)) \hookrightarrow \Gamma_c(\mathcal{B}) \hookrightarrow C^*(\mathcal{B}),$$

and since  $E(\Gamma_c(\mathcal{B} * \Omega))$  is dense in  $C^*(\mathcal{B} * \Omega)^\alpha$ , we have  $\Phi(C^*(\mathcal{B})) = C^*(\mathcal{B} * \Omega)^\alpha$ .  $\square$

*Proof of Theorem 3.2.* We will show that the identity map on  $\Gamma_c(\mathcal{B} * T)$  is compatible with the regular representation  $\Lambda$  and the homomorphism  $\Phi$  from Proposition 2.1.

We begin by recalling the formulas associated with the imprimitivity bimodules  $X$  and  $X_R$ . For  $X$ , first recall the abstract formulas from [7] for the imprimitivity bimodule associated to an equivalence bundle  $\mathcal{E} \rightarrow \Omega$  between Fell bundles  $\mathcal{B} \rightarrow \mathcal{G}$  and  $\mathcal{C} \rightarrow \mathcal{H}$ :

$$\begin{aligned} (f \cdot \xi)(t) &= \int_{\mathcal{G}} f(x) \cdot \xi(x^{-1} \cdot t) d\lambda_{\mathcal{G}}^{\rho(t)}(x) \\ {}_L\langle \xi, \eta \rangle(x) &= \int_{\mathcal{H}} {}_{\mathcal{B}}\langle \xi(x \cdot t \cdot h), \eta(t \cdot h) \rangle d\lambda_{\mathcal{H}}^{\sigma(t)}(h) \\ (\xi \cdot f)(t) &= \int_{\mathcal{H}} \xi(t \cdot h) \cdot f(h^{-1}) d\lambda_{\mathcal{H}}^{\sigma(t)}(h) \\ \langle \xi, \eta \rangle_R(h) &= \int_{\mathcal{G}} \langle \xi(x^{-1} \cdot t), \eta(x^{-1} \cdot t \cdot h) \rangle_{\mathcal{C}} d\lambda^{\rho(t)}(x), \end{aligned}$$

where in the second and fourth equations  $t \in \Omega$  is any element satisfying  $\rho(t) = s(x)$  and  $\sigma(t) = r(h)$ , respectively.

In our context, we have an equivalence  $\mathcal{B} * \Omega \rightarrow \mathcal{Y} * \Omega$  between Fell bundles  $(\mathcal{B} * \Omega) \rtimes G \rightarrow (\mathcal{Y} * \Omega) \rtimes G$  and  $\mathcal{B} \rightarrow \mathcal{Y}$ . The left module action becomes

$$\begin{aligned} (f \cdot \xi)(y, u) &= \int_{(\mathcal{Y} * \Omega) \rtimes G} f(x, v, s) \cdot \xi((x, v, s)^{-1} \cdot (y, u)) d\lambda_{(\mathcal{Y} * \Omega) \rtimes G}^{\rho(y, u)}(x, v, s) \\ &= \int_{(\mathcal{Y} * \Omega) \rtimes G} f(x, v, s) \cdot \xi\left(s^{-1} \cdot (x, v)^{-1}, s^{-1}\right) \cdot (y, u) d\lambda_{(\mathcal{Y} * \Omega) \rtimes G}^{(r(y), y \cdot u, e)}(x, v, s) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathcal{Y} * \Omega} \int_G f(x, v, s) \cdot \xi\left((s^{-1} \cdot (x^{-1}, x \cdot v), s^{-1}) \cdot (y, u)\right) ds d\lambda_{\mathcal{Y} * \Omega}^{(r(y), y \cdot u)}(x, v) \\
&= \int_{\mathcal{Y}} \int_G f(x, x^{-1}y \cdot u, s) \cdot \xi((x^{-1}, s^{-1} \cdot x \cdot v, s^{-1}) \cdot (y, u)) ds d\lambda_{\mathcal{Y}}^{r(y)}(x) \\
&\quad \text{because we must have } y \cdot u = x \cdot v \\
&= \int_{\mathcal{Y}} \int_G f(x, x^{-1}y \cdot u, s) \cdot \xi(x^{-1}y, s^{-1} \cdot u) ds d\lambda_{\mathcal{Y}}^{r(y)}(x).
\end{aligned}$$

The left inner product becomes

$$\begin{aligned}
&{}_L \langle \xi, \eta \rangle(y, u, s) \\
&= \int_{\mathcal{Y}} (\mathcal{B} * \Omega) \rtimes G \left\langle \xi((y, u, s) \cdot (x, v) \cdot z), \eta((x, v) \cdot z) \right\rangle d\lambda^{\sigma(x, v)}(z) \\
&\quad \text{where } \rho(x, v) = s(y, u, s) \\
&= \int_{\mathcal{Y}} (\mathcal{B} * \Omega) \rtimes G \left\langle \xi((yx, s \cdot v) \cdot z), \eta(xz, z^{-1} \cdot v) \right\rangle d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} (\mathcal{B} * \Omega) \rtimes G \left\langle \xi(yxz, z^{-1} \cdot s \cdot v), \eta(xz, z^{-1} \cdot v) \right\rangle d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} (\mathcal{B} * \Omega) \rtimes G \left\langle \xi(yxz, s \cdot z^{-1} \cdot v), \eta(xz, z^{-1} \cdot v) \right\rangle d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} (\mathcal{B} * \Omega) \rtimes G \left\langle (\xi_1(yxz, s \cdot z^{-1} \cdot v), x \cdot z^{-1} \cdot v), \right. \\
&\quad \left. (\eta_1(xz, z^{-1} \cdot v), z^{-1} \cdot v) \right\rangle d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} (\xi_1(yxz, s \cdot z^{-1} \cdot v) \eta_1(xz, z^{-1} \cdot v)^*, \\
&\quad s \cdot p(\eta_1(xz, z^{-1} \cdot v)) \cdot z^{-1} \cdot v, s) d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} (\xi_1(yxz, s \cdot z^{-1} \cdot v) \eta_1(xz, z^{-1} \cdot v)^*, s \cdot xz \cdot z^{-1} \cdot v, s) d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} (\xi_1(yxz, z^{-1}x^{-1} \cdot u) \eta_1(xz, s^{-1} \cdot z^{-1}x^{-1} \cdot u)^*, u, s) d\lambda^{s(x)}(z)
\end{aligned}$$

because  $\rho(x, v) = (r(x), x \cdot v, e)$  and  $s(y, u, s) = (s(y), s^{-1} \cdot u, e)$ , and thus

$$\begin{aligned}
&(\mathcal{B} * \Omega) \rtimes G \langle \xi, \eta \rangle_1(y, u, s) \\
&= \int_{\mathcal{Y}} \xi_1(yxz, z^{-1}x^{-1} \cdot u) \eta_1(xz, s^{-1} \cdot z^{-1}x^{-1} \cdot u)^* d\lambda^{s(x)}(z) \\
&= \int_{\mathcal{Y}} \xi_1(yz, z^{-1} \cdot u) \eta_1(z, s^{-1} \cdot z^{-1} \cdot u)^* d\lambda^{r(x)}(z) \\
&= \int_{\mathcal{Y}} \xi_1(yz, z^{-1} \cdot u) \eta_1(z, s^{-1} \cdot z^{-1} \cdot u)^* d\lambda^{s(y)}(z).
\end{aligned}$$

The right module action becomes

$$\begin{aligned}
(\xi \cdot f)(y, u) &= \int_{\mathcal{Y}} \xi((y, u) \cdot x) \cdot f(x^{-1}) d\lambda_{\mathcal{Y}}^{\sigma(y, u)}(x) \\
&= \int_{\mathcal{Y}} \xi(yx, x^{-1} \cdot u) \cdot f(x^{-1}) d\lambda_{\mathcal{Y}}^{s(y)}(x),
\end{aligned}$$

so

$$(\xi \cdot f)_1(y, u) = \int_{\mathcal{Y}} \xi_1(yx, x^{-1} \cdot u) f(x^{-1}) d\lambda^{s(y)}(x).$$

The right inner product becomes

$$\begin{aligned} \langle \xi, \eta \rangle_R(y) &= \int_{(\mathcal{Y} * \Omega) \rtimes G} \left\langle \xi((x, u, s)^{-1} \cdot (z, v)), \right. \\ &\quad \left. \eta((x, u, s)^{-1} \cdot (z, v) \cdot y) \right\rangle_{\mathcal{B}} d\lambda^{\rho(z, v)}(x, u, s), \end{aligned}$$

where  $(z, v)$  is any element of  $\mathcal{Y} * \Omega$  such that  $\sigma(z, v) = r(y)$ . Since  $\sigma(z, v) = s(z)$ , we can take  $z = y^{-1}$ . We have

$$\begin{aligned} (x, u, s)^{-1} &= (s^{-1} \cdot (x, u)^{-1}, s^{-1}) \\ &= (s^{-1} \cdot (x^{-1}, x \cdot u), s^{-1}) \\ &= (x^{-1}, s^{-1} \cdot x \cdot u, s^{-1}), \\ (x^{-1}, s^{-1} \cdot x \cdot u, s^{-1}) \cdot (y^{-1}, v) &= (x^{-1}y^{-1}, s^{-1} \cdot v), \\ (x^{-1}y^{-1}, s^{-1} \cdot v) \cdot y &= (x^{-1}, y^{-1} \cdot s^{-1} \cdot v), \end{aligned}$$

and

$$\rho(y^{-1}, v) = (s(y), y^{-1} \cdot v, e),$$

so we get

$$\begin{aligned} &\langle \xi, \eta \rangle_R(y) \\ &= \int_{(\mathcal{Y} * \Omega) \rtimes G} \left\langle \xi(x^{-1}y^{-1}, s^{-1} \cdot v), \right. \\ &\quad \left. \eta(x^{-1}, y^{-1} \cdot s^{-1} \cdot v) \right\rangle_{\mathcal{B}} d\lambda^{(s(y), y^{-1} \cdot v, e)}(x, u, s) \\ &\quad \text{where } y^{-1} \cdot v = x \cdot u \\ &= \int_{\mathcal{Y}} \int_G \left\langle (\xi_1(x^{-1}y^{-1}, s^{-1} \cdot v), s^{-1} \cdot v), \right. \\ &\quad \left. (\eta(x^{-1}, y^{-1} \cdot s^{-1} \cdot v), y^{-1} \cdot s^{-1} \cdot v) \right\rangle_{\mathcal{B}} ds d\lambda^{s(y)}(x) \\ &= \int_{\mathcal{Y}} \int_G \xi_1(x^{-1}y^{-1}, s^{-1} \cdot v)^* \eta(x^{-1}, y^{-1} \cdot s^{-1} \cdot v) ds d\lambda^{s(y)}(x) \\ &= \int_{\mathcal{Y}} \int_G \xi_1(x^{-1}y^{-1}, s^{-1} \cdot yx \cdot u)^* \eta(x^{-1}, s^{-1} \cdot x \cdot u) ds d\lambda^{s(y)}(x). \end{aligned}$$

For  $X_R$ , recall the abstract formulas from [9] for the imprimitivity bimodule associated to an action  $\alpha : G \rightarrow \text{Aut } A$  (where  $A$  is any  $C^*$ -algebra) that is saturated and proper with respect to a dense  $*$ -subalgebra  $A_0$ : for  $f \in C_c(G, A) \subset A \rtimes_{\alpha, r} G$  and  $\xi, \eta, \zeta \in A_0$  we have

$$\begin{aligned} f \cdot \xi &= \int_G f(s) \alpha_s(\xi) ds \\ {}_{A \rtimes_{\alpha, r} G} \langle \xi, \eta \rangle(s) &= \Delta(s)^{-1/2} \xi \alpha_s(\eta^*) \\ \zeta \langle \xi, \eta \rangle_{A^\alpha} &= \int_G \zeta \alpha_s(\xi^* \eta) ds, \end{aligned}$$

and of course the right module action of  $A^\alpha$  on  $A$  is given by right multiplication.

In our context we have  $A = C^*(\mathcal{B} * \Omega)$  and  $A_0 = \Gamma_c(\mathcal{B} * \Omega)$ . We take

$$f \in C_c(G, \Gamma_c(\mathcal{B} * \Omega)) \subset \Gamma_c((\mathcal{B} * \Omega) \rtimes G),$$

and then the left module action becomes

$$\begin{aligned} (f \cdot \xi)(y, u) &= \int_G (f(s) \alpha_s(\xi))(y, u) ds \\ &= \int_G \int_{\mathcal{Y} * \Omega} f(s)(x, v) \alpha_s(\xi)((x, v)^{-1}(y, u)) d\lambda_{\mathcal{Y}}^{r(y, u)}(x, v) ds \\ &= \int_G \int_{\mathcal{Y}} f(s)(x, v) \alpha_s(\xi)((x^{-1}, x \cdot v)(y, u)) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} f(s)(x, v) \alpha_s(\xi)(x^{-1}y, u) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} f(s)(x, v) s \cdot \left( \xi(s^{-1} \cdot (x^{-1}, x \cdot v)) \right) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} f(s)(x, v) s \cdot \left( \xi(x^{-1}, s^{-1} \cdot x \cdot v) \right) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} f(s)(x, v) s \cdot \left( \xi_1(x^{-1}, s^{-1} \cdot x \cdot v), s^{-1} \cdot x \cdot v \right) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} (f(s)_1(x, v), v) \left( \xi_1(x^{-1}, s^{-1} \cdot x \cdot v), x \cdot v \right) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} (f_1(x, v, s) \xi_1(x^{-1}, s^{-1} \cdot x \cdot v), x \cdot v) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds, \end{aligned}$$

where the meaning of the notation  $f_1$  is clear once we identify  $f \in C_c(G, \Gamma_c(\mathcal{B} * \Omega))$  with the corresponding section in  $\Gamma_c((\mathcal{B} * \Omega) \rtimes G)$ , and thus

$$\begin{aligned} (f \cdot \xi)_1(y, u) &= \int_G \int_{\mathcal{Y}} f_1(x, v, s) \xi_1(x^{-1}, s^{-1} \cdot x \cdot v) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} f_1(x, x^{-1}y \cdot u, s) \xi_1(x^{-1}, s^{-1} \cdot y \cdot u) d\lambda_{\mathcal{Y}}^{r(y)}(x) ds, \end{aligned}$$

because  $y \cdot u = x \cdot v$ .

For the left inner product on  $\Gamma_c(\mathcal{B} * \Omega) \subset X_R$ , if  $\xi, \eta \in \Gamma_c(\mathcal{B} * \Omega)$  then the inner product  ${}_{C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G} \langle \xi, \eta \rangle$  lies in  $C_c(G, \Gamma_c(\mathcal{B} * \Omega)) \subset \Gamma_c((\mathcal{B} * \Omega) \rtimes G)$ , and we have

$$\begin{aligned} {}_{C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G} \langle \xi, \eta \rangle(y, u, s) &= {}_{C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G} \langle \xi, \eta \rangle(s)(y, u) \\ &= \Delta(s)^{-1/2} (\xi \alpha_s(\eta^*)) (y, u) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y} * \Omega} \xi(x, v) \alpha_s(\eta^*)((x, v)^{-1}(y, u)) d\lambda_{\mathcal{Y} * \Omega}^{r(y, u)}(x, v) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y} * \Omega} \xi(x, v) \alpha_s(\eta^*)((x^{-1}, x \cdot v)(y, u)) d\lambda_{\mathcal{Y} * \Omega}^{r(y), y \cdot v}(x, v) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y}} \xi(x, v) \alpha_s(\eta^*)(x^{-1}y, u) d\lambda_{\mathcal{Y}}^{r(y)}(x) \end{aligned}$$

$$\begin{aligned}
&= \Delta(s)^{-1/2} \int_{\mathcal{Y}} (\xi_1(x, v), v) (\eta_1^*(x^{-1}y, s^{-1} \cdot u), u) d\lambda_{\mathcal{Y}}^{r(y)}(x) \\
&= \Delta(s)^{-1/2} \int_{\mathcal{Y}} (\xi_1(x, v), v) (\eta_1(y^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^*, u) d\lambda_{\mathcal{Y}}^{r(y)}(x)
\end{aligned}$$

so

$$\begin{aligned}
&C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r, G} \langle \xi, \eta \rangle_1(y, u, s) \\
&= \Delta(s)^{-1/2} \int_{\mathcal{Y}} \xi_1(x, x^{-1}y \cdot u) \eta_1(y^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^* d\lambda_{\mathcal{Y}}^{r(y)}(x),
\end{aligned}$$

because  $y \cdot u = x \cdot v$ .

For the right inner product, if  $\xi, \eta, \zeta \in \Gamma_c(\mathcal{B} * \Omega)$  then

$$\begin{aligned}
&(\zeta \langle \xi, \eta \rangle_{C^*(\mathcal{B} * \Omega)^\alpha})(y, u) \\
&= \int_G (\zeta \alpha_s(\xi^* \eta))(y, u) ds \\
&= \int_G \int_{\mathcal{Y} * \Omega} \zeta(x, v) \alpha_s(\xi^* \eta)((x, v)^{-1}(y, u)) d\lambda^{r(y, u)}(x, v) ds \\
&= \int_G \int_{\mathcal{Y} * \Omega} \zeta(x, v) \alpha_s(\xi^* \eta)(x^{-1}y, u) d\lambda^{(r(y), y \cdot u)}(x, v) ds \\
&= \int_G \int_{\mathcal{Y}} (\zeta_1(x, v), v) ((\xi^* \eta)_1(x^{-1}y, s^{-1} \cdot u), u) d\lambda^{r(y)}(x) ds,
\end{aligned}$$

so

$$\begin{aligned}
&(\zeta \langle \xi, \eta \rangle_{C^*(\mathcal{B} * \Omega)^\alpha})_1(y, u) \\
&= \int_G \int_{\mathcal{Y}} \zeta_1(x, v) (\xi^* \eta)_1(x^{-1}y, s^{-1} \cdot u) d\lambda^{r(y)}(x) ds \\
&= \int_G \int_{\mathcal{Y}} \zeta_1(x, v) \int_{\mathcal{Y}} \xi_1(z^{-1}, s^{-1} \cdot x^{-1}y \cdot u)^* \eta_1(z^{-1}x^{-1}y, s^{-1} \cdot u) d\lambda^{r(x^{-1}y)}(z) \\
&\quad d\lambda^{r(y)}(x) ds \\
&= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \xi_1(z^{-1}, s^{-1} \cdot x^{-1}y \cdot u)^* \eta_1(z^{-1}x^{-1}y, s^{-1} \cdot u) \\
&\quad d\lambda^{s(x)}(z) d\lambda^{r(y)}(x) ds.
\end{aligned}$$

We will need to observe the following: for  $f \in \Gamma_c(\mathcal{B})$  and  $\xi \in \Gamma_c(\mathcal{B} * \Omega)$  we have

$$\begin{aligned}
&(\xi \Phi(f))_1(y, u) = (\Phi(f^*) \xi^*)_1^*(y, u) \\
&= (\Phi(f^*) \xi^*)_1(y^{-1}, y \cdot u)^* \\
&= \left( \int_{\mathcal{Y}} f^*(x) \xi_1^*(x^{-1}y^{-1}, y \cdot u) d\lambda^{r(y^{-1})}(x) \right)^* \\
&= \left( \int_{\mathcal{Y}} f(x^{-1})^* \xi_1(yx, x^{-1} \cdot u)^* d\lambda^{s(y)}(x) \right)^* \\
&= \int_{\mathcal{Y}} \xi_1(yx, x^{-1} \cdot u) f(x^{-1}) d\lambda^{s(y)}(x).
\end{aligned}$$

We now proceed to show that the identity map on  $\Gamma_c(\mathcal{B} * \Omega)$  extends to an imprimitivity-bimodule map  $\Upsilon : X \rightarrow X_R$  (which will then be a surjection because  $\Gamma_c(\mathcal{B} * \Omega)$  is dense in  $X_R$ ). It suffices to show that, on generators in  $\Gamma_c(\mathcal{B} * \Omega)$ , the

maps  $\Lambda$  and  $\Phi$  transport the inner products of the imprimitivity bimodule  $X$  to those of  $X_R$ . That is, we must show that for  $\xi, \eta \in \Gamma_c(\mathcal{B} * \Omega)$  we have

$$(3.2) \quad \Lambda(C^*(\mathcal{B} * \Omega) \rtimes_{\alpha} G \langle \xi, \eta \rangle) = C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G \langle \xi, \eta \rangle;$$

$$(3.3) \quad \Phi(\langle \xi, \eta \rangle_{C^*(\mathcal{B})}) = \langle \xi, \eta \rangle_{C^*(\mathcal{B} * \Omega)^{\alpha}}.$$

For (3.2), first of all we have

$$\begin{aligned} & C^*((\mathcal{B} * \Omega) \rtimes G) \langle \xi, \eta \rangle_1(y, u, s) \\ &= \int_{\mathcal{Y}} \xi_1(yz, z^{-1} \cdot u) \eta_1(z, s^{-1} \cdot z^{-1} \cdot u)^* d\lambda^{s(y)}(z) \\ &= \int_{\mathcal{Y}} \xi_1(z, x^{-1}y \cdot u) \eta_1(y^{-1}x, s^{-1} \cdot x^{-1}y^{-1} \cdot u)^* d\lambda^{r(y)}(x) \\ & \quad \text{after substituting } x = yz. \end{aligned}$$

Recall from [5, Theorem 7.1] that we have an isomorphism

$$C^*(\mathcal{B} * \Omega) \rtimes_{\alpha} G \cong C^*((\mathcal{B} * \Omega) \rtimes G),$$

and we will actually blur the distinction between these two  $C^*$ -algebras, so that for  $f \in \Gamma_c(\mathcal{B} * \Omega)$  and  $g \in C_c(G)$  the generator  $i_{C^*(\mathcal{B} * \Omega)}(f)i_G(g)$  of the crossed product  $C^*(\mathcal{B} * \Omega) \rtimes_{\alpha} G$  is identified with the element of  $\Gamma_c((\mathcal{B} * \Omega) \rtimes G)$  given by

$$(i_{C^*(\mathcal{B} * \Omega)}(f)i_G(g))(x, u, t) = (f_1(x, u)g(t)\Delta(t)^{1/2}, u, t).$$

Thus  $C^*((\mathcal{B} * \Omega) \rtimes_{\alpha} G) \langle \xi, \eta \rangle$  is the element of

$$\begin{aligned} C_c(G, \Gamma_c(\mathcal{B} * \Omega)) &\subset C_c(G, C^*(\mathcal{B} * \Omega)) \\ &\subset C^*(\mathcal{B} * \Omega) \rtimes_{\alpha} G \end{aligned}$$

satisfying

$$\begin{aligned} & C^*((\mathcal{B} * \Omega) \rtimes G) \langle \xi, \eta \rangle(s)_1(y, u) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y}} \xi_1(x, x^{-1}y \cdot u) \eta_1(y^{-1}x, s^{-1} \cdot x^{-1}y^{-1} \cdot u)^* d\lambda^{r(y)}(x). \end{aligned}$$

On the other hand, we have

$$C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G \langle \xi, \eta \rangle(s) = \Delta(s)^{-1/2} (\xi \alpha_s(\eta)^*),$$

so  $C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G \langle \xi, \eta \rangle$  is the element of

$$\begin{aligned} C_c(G, \Gamma_c(\mathcal{B} * \Omega)) &\subset C_c(G, C^*(\mathcal{B} * \Omega)) \\ &\subset C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G \end{aligned}$$

satisfying

$$\begin{aligned} & C^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G \langle \xi, \eta \rangle(s)_1(y, u) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y}} \xi_1(x, x^{-1}y \cdot u) \alpha_s(\eta)_1^*(x^{-1}y, u) d\lambda^{r(y)}(x) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y}} \xi_1(x, x^{-1}y \cdot u) \alpha_s(\eta)_1(y^{-1}x, x^{-1}y \cdot u)^* d\lambda^{r(y)}(x) \\ &= \Delta(s)^{-1/2} \int_{\mathcal{Y}} \xi_1(x, x^{-1}y \cdot u) \eta_1(y^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^* d\lambda^{r(y)}(x). \end{aligned}$$

Therefore, since  $\Lambda$  is the bounded extension of the identity map on  $C_c(G, C^*(\mathcal{B} * \Omega))$ , we have verified (3.2).

For (3.3), we will actually find it convenient to show that if  $\zeta \in \Gamma_c(\mathcal{B} * \Omega)$ , then

$$\zeta \Phi(\langle \xi, \eta \rangle_{C^*(\mathcal{B})}) = \zeta \langle \xi, \eta \rangle_{C^*(\mathcal{B} * \Omega)^\alpha}.$$

The left side is the element of  $\Gamma_c(\mathcal{B} * \Omega)$  satisfying

$$\begin{aligned} & \left( \zeta \Phi(\langle \xi, \eta \rangle_{C^*(\mathcal{B})}) \right)_1(y, u) \\ &= \int_{\mathcal{Y}} \zeta_1(yx, x^{-1} \cdot u) \langle \xi, \eta \rangle_{C^*(\mathcal{B})}(x^{-1}) d\lambda^{s(y)}(x) \\ &= \int_{\mathcal{Y}} \zeta_1(yx, x^{-1} \cdot u) \int_{\mathcal{Y}} \int_G \xi_1(z^{-1}x, s^{-1} \cdot x^{-1}z \cdot w)^* \\ & \quad \eta_1(z^{-1}, s^{-1} \cdot z \cdot w) ds d\lambda^{s(x^{-1})}(z) d\lambda^{s(y)}(x) \\ & \quad \text{where } \rho(w) = s(z); \text{ can take } w = z^{-1} \cdot u \\ &= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(yx, x^{-1} \cdot u) \xi_1(z^{-1}x, s^{-1} \cdot x^{-1} \cdot u)^* \\ & \quad \eta_1(z^{-1}, s^{-1} \cdot u) d\lambda^{r(x)}(z) d\lambda^{s(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(yx, x^{-1} \cdot u) \xi_1(z^{-1}x, s^{-1} \cdot x^{-1} \cdot u)^* \\ & \quad \eta_1(z^{-1}, s^{-1} \cdot u) d\lambda^{s(y)}(z) d\lambda^{s(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \xi_1(z^{-1}y^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^* \\ & \quad \eta_1(z^{-1}, s^{-1} \cdot u) d\lambda^{s(y)}(z) d\lambda^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \xi_1(z^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^* \\ & \quad \eta_1(z^{-1}y, s^{-1} \cdot u) d\lambda^{r(y)}(z) d\lambda^{r(y)}(x) ds. \end{aligned}$$

On the other hand,  $\zeta \langle \xi, \eta \rangle_{C^*(\mathcal{B} * \Omega)^\alpha}$  is the element of  $\Gamma_c(\mathcal{B} * \Omega)$  satisfying

$$\begin{aligned} & (\zeta \langle \xi, \eta \rangle_{C^*(\mathcal{B} * \Omega)^\alpha})_1(y, u) \\ &= \int_G (\zeta \alpha_s(\xi^* \eta))_1(y, u) ds \\ &= \int_G \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) (\alpha_s(\xi^* \eta))_1(x^{-1}y, u) d\lambda^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) (\xi^* \eta)_1(x^{-1}y, s^{-1} \cdot u) d\lambda^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \int_{\mathcal{Y}} \xi_1^*(z, z^{-1}x^{-1}y \cdot s^{-1} \cdot u) \\ & \quad \eta_1(z^{-1}x^{-1}y, s^{-1} \cdot u) d\lambda^{r(x^{-1}y)}(z) d\lambda^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \xi_1(z^{-1}, s^{-1} \cdot x^{-1}y \cdot u)^* \\ & \quad \eta_1(z^{-1}x^{-1}y, s^{-1} \cdot u) d\lambda^{s(x)}(z) d\lambda^{r(y)}(x) ds \\ &= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \xi_1(z^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^* \end{aligned}$$

$$\begin{aligned}
& \eta_1(z^{-1}y, s^{-1} \cdot u) d\lambda^{r(x)}(z) d\lambda^{r(y)}(x) ds \\
&= \int_G \int_{\mathcal{Y}} \int_{\mathcal{Y}} \zeta_1(x, x^{-1}y \cdot u) \xi_1(z^{-1}x, s^{-1} \cdot x^{-1}y \cdot u)^* \\
& \quad \eta_1(z^{-1}y, s^{-1} \cdot u) d\lambda^{r(y)}(z) d\lambda^{r(y)}(x) ds.
\end{aligned}$$

Therefore (3.3) holds, and we are done.  $\square$

Using a recent result of Sims and Williams, we can show that in Theorem 3.2 the surjection of  $C^*(\mathcal{B})$  onto the generalized fixed-point algebra can be identified with the regular representation:

**Corollary 3.4.** *Let  $p : \mathcal{B} \rightarrow \mathcal{Y}$  be a Fell bundle over a locally compact groupoid, and let  $G$  be a locally compact group. Suppose that both  $\mathcal{Y}$  and  $G$  act on (the left of) a locally compact Hausdorff space  $\Omega$ , and that the action of  $G$  is free and proper and commutes with the  $\mathcal{Y}$ -action, so that  $G$  also acts freely and properly by automorphisms on the transformation Fell bundle  $\mathcal{B} * \Omega \rightarrow \mathcal{Y} * \Omega$ , and we also have an associated action  $\alpha : G \rightarrow \text{Aut } C^*(\mathcal{B} * \Omega)$ . Then there is a unique isomorphism  $\Xi$  making the diagram*

$$(3.4) \quad \begin{array}{ccc} & C^*(\mathcal{B}) & \\ \Phi \swarrow & & \searrow \Lambda \\ C^*(\mathcal{B} * \Omega)^\alpha & \xrightarrow[\Xi]{\cong} & C_r^*(\mathcal{B}) \end{array}$$

commute.

*Proof.* By [11, Theorem 14] the kernels of the regular representations of  $C^*((\mathcal{B} * \Omega) \times G)$  and  $C^*(\mathcal{B})$  correspond via the imprimitivity bimodule  $X$ . By [5, Theorem 7.1] we have

$$C^*((\mathcal{B} * \Omega) \times G) \cong C^*(\mathcal{B} * \Omega) \rtimes_\alpha G,$$

by [11, Example 11] we have

$$C_r^*((\mathcal{B} * \Omega) \times G) \cong C_r^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G,$$

and the regular representations

$$\begin{aligned}
\Lambda : C^*((\mathcal{B} * \Omega) \times G) &\rightarrow C_r^*((\mathcal{B} * \Omega) \times G) \\
\Lambda : C^*(\mathcal{B} * \Omega) \rtimes_\alpha G &\rightarrow C_r^*(\mathcal{B} * \Omega) \rtimes_{\alpha, r} G
\end{aligned}$$

correspond under these isomorphisms. Thus the kernels of the regular representations of  $C^*(\mathcal{B} * \Omega) \rtimes_\alpha G$  and  $C^*(\mathcal{B})$  correspond via  $X$ . But by Theorem 3.2 the kernels of the regular representation of  $C^*(\mathcal{B} * \Omega) \rtimes_\alpha G$  and of  $\Phi : C^*(\mathcal{B}) \rightarrow C^*(\mathcal{B} * \Omega)^\alpha$  also correspond via  $X$ , so the result follows.  $\square$

It is convenient to have the following alternative version of Corollary 3.4:

**Corollary 3.5.** *Let  $p : \mathcal{A} \rightarrow \mathcal{X}$  be a Fell bundle over a locally compact groupoid, and let  $G$  be a locally compact group. Suppose that  $G$  acts freely and properly on (the left of)  $\mathcal{A}$  by automorphisms, so that we also have an associated action*



$\alpha : G \rightarrow \text{Aut } C^*(\mathcal{A})$ . Then there is a unique isomorphism  $\Xi$  making the diagram

$$(3.5) \quad \begin{array}{ccc} & C^*(G \setminus \mathcal{A}) & \\ \Phi \swarrow & & \searrow \Lambda \\ C^*(\mathcal{A})^\alpha & \xrightarrow[\Xi]{\cong} & C_r^*(G \setminus \mathcal{A}) \end{array}$$

commute.

#### 4. APPLICATIONS

**4.1. Coaction-crossed products.** Let  $\mathcal{B} \rightarrow G$  be a Fell bundle over a locally compact group. Then by [5, Theorem 5.1] we have an equivariant isomorphism

$$(C^*(\mathcal{B} \times G), \alpha) \cong (C^*(\mathcal{B}) \rtimes_\delta G, \widehat{\delta}),$$

so the Rieffel Surjection Theorem 3.2 in this context can be expressed in the form

$$(\Lambda, \Upsilon, \Phi) : (C^*(\mathcal{B}) \rtimes_\delta G \rtimes_{\widehat{\delta}} G, X, C^*(\mathcal{B})) \rightarrow (C^*(\mathcal{B}) \rtimes_\delta G \rtimes_{\widehat{\delta}, r} G, X_R, C_r^*(\mathcal{B}))$$

Moreover, in this case we can identify the isomorphism

$$\Xi : (C^*(\mathcal{B}) \rtimes_\delta G)^{\widehat{\delta}} \xrightarrow[\cong]{} C_r^*(\mathcal{B})$$

of (3.4): the Banach bundle  $\mathcal{B} \rightarrow G$  gives an equivalence between the Fell bundles  $\mathcal{B} \times G \rightarrow G \times G$  and  $B(e) \rightarrow \{e\}$ , and hence by the YMW Theorem we have a  $C^*(\mathcal{C}) \rtimes_\delta G - B(e)$  imprimitivity bimodule  $L^2(\mathcal{B})$ , and hence an isomorphism

$$C^*(\mathcal{B}) \rtimes_\delta G \xrightarrow[\cong]{\varphi} \mathcal{K}(L^2(\mathcal{B})).$$

**Theorem 4.1.** *With the above notation, the isomorphism  $\Xi$  of (3.4) is the restriction to  $(C^*(\mathcal{B}) \rtimes_\delta G)^{\widehat{\delta}}$  of the canonical extension*

$$\overline{\varphi} : M(C^*(\mathcal{B}) \rtimes_\delta G) \rightarrow \mathcal{L}(L^2(\mathcal{B})).$$

*Proof.* Let

$$C^*(\mathcal{B}) \rtimes_\delta G \xrightarrow[\theta]{\cong} C^*(\mathcal{B} \times G)$$

be the isomorphism of [5, Theorem 5.1], and let

$$\psi = \varphi \circ \theta^{-1} : C^*(\mathcal{B} \times G) \rightarrow \mathcal{L}(L^2(\mathcal{B})).$$

Since  $\Phi(C^*(\mathcal{B})) = C^*(\mathcal{B} \times G)^\alpha$  and  $\Lambda(C^*(\mathcal{B})) = C_r^*(\mathcal{B})$ , it suffices to show that the diagram

$$(4.1) \quad \begin{array}{ccc} & C^*(\mathcal{B}) & \\ \Phi \swarrow & & \searrow \Lambda \\ M(C^*(\mathcal{B} \times G)) & \xrightarrow[\overline{\psi}]{} & \mathcal{L}(L^2(\mathcal{B})) \end{array}$$

commutes. Let's recall that for  $g \in \Gamma_c(\mathcal{B} \times G)$  and  $\xi \in \Gamma_c(\mathcal{B})$  we have

$$\begin{aligned} (\psi(g)\xi)(x) &= \int_{G \rtimes_{\text{lt}} G} g_1(y, u) \xi((y, u)^{-1} \cdot x) d\lambda_{G \rtimes_{\text{lt}} G}^{r(x)}(y, u) \\ &= \int_G g_1(y, y^{-1}x) \xi(y^{-1}x) dy. \end{aligned}$$

It suffices to check commutativity of the diagram on functions  $f \in \Gamma_c(\mathcal{B})$ , and it suffices to check the values of  $\bar{\psi} \circ \Phi(f)$  and  $\Lambda(f)$  on vectors in  $\ell^2(\mathcal{B})$  of the form  $\psi(g)\xi$  for  $g \in \Gamma_c(\mathcal{B} \times G)$  and  $\xi \in \ell^2(\mathcal{B})$ :

$$\begin{aligned}
(\bar{\psi}(\Phi(f))\psi(g)\xi)(x) &= (\psi(\Phi(f)g)\xi)(x) \\
&= \int_G (\Phi(f)g)_1(y, y^{-1}x) \xi(y^{-1}x) dy \\
&= \int_G \int_G f(s) g_1(s^{-1}y, y^{-1}x) ds \xi(y^{-1}x) dy \\
&= \int_G \int_G f(s) g_1(s^{-1}y, y^{-1}x) \xi(y^{-1}x) dy ds \\
&= \int_G f(s) \int_G g_1(y, y^{-1}s^{-1}x) \xi(y^{-1}s^{-1}x) dy ds \\
&= \int_G f(s) (\psi(g)\xi)(s^{-1}x) ds \\
&= (\Lambda(f)\psi(g)\xi)(x).
\end{aligned}$$

Thus (4.1) commutes.  $\square$

We can deduce from the above that, as one would expect, the regular representation of  $C^*(\mathcal{B})$  is a normalization:

**Corollary 4.2.** *Let  $\mathcal{B} \rightarrow G$  be a Fell bundle over a locally compact group, and let  $\delta$  be the canonical coaction of  $G$  on  $C^*(\mathcal{B})$ . Then there is a unique coaction  $\delta^n$  of  $G$  on  $C_r^*(\mathcal{B})$  such that the regular representation*

$$(C^*(\mathcal{B}), \delta) \xrightarrow{\Lambda} (C_r^*(\mathcal{B}), \delta^n).$$

*is a normalization of  $\delta$ .*

*Proof.* We will show that the diagram

$$(4.2) \quad \begin{array}{ccc} & C^*(\mathcal{B}) & \\ j_{\mathcal{B}} \swarrow & & \searrow \Phi \\ M(C^*(\mathcal{B}) \rtimes_{\delta} G) & \xrightarrow[\bar{\theta}]{\cong} & M(C^*(\mathcal{B} \times G)) \end{array}$$

commutes. It will then follow from Theorem 4.1 that the diagram

$$\begin{array}{ccc} & C^*(\mathcal{B}) & \\ j_{\mathcal{B}} \swarrow & & \searrow \Lambda \\ j_{\mathcal{B}}(C^*(\mathcal{B})) & \xrightarrow[\Xi \circ \theta]{\cong} & C_r^*(\mathcal{B}) \end{array}$$

commutes. Since there is a unique coaction  $\text{Ad } j_G$  on  $j_{\mathcal{B}}(C^*(\mathcal{B}))$  such that

$$j_{\mathcal{B}} : (C^*(\mathcal{B}), \delta) \rightarrow (j_{\mathcal{B}}(C^*(\mathcal{B})), \text{Ad } j_G)$$

is a normalization, this will complete the proof. The following computation implies that (4.2) commutes: for  $f, g \in \Gamma_c(\mathcal{B})$ ,  $h \in C_c(G)$ , and  $k \in \Gamma_c(\mathcal{B} \times G)$  we have

$$\left( \bar{\theta} \circ j_{\mathcal{B}}(f) \left( \theta(j_{\mathcal{B}}(g)j_G(h)) * k \right) \right)(s, t)$$

$$\begin{aligned}
&= \left( \theta(j_{\mathcal{B}}(f)j_{\mathcal{B}}(g)j_G(h)) * k \right)(s, t) \\
&= \left( \theta(j_{\mathcal{B}}(f * g)j_G(h)) * k \right)(s, t) \\
&= \left( (\Delta^{1/2}(f * g) \boxtimes h) * k \right)(s, t) \\
&= \int_G (\Delta^{1/2}(f * g) \boxtimes h)(r, r^{-1}st) k(r^{-1}s, t) dr \\
&= \int_G \Delta(r)^{1/2} (f * g)(r) h(r^{-1}st) k(r^{-1}s, t) dr \\
&= \int_G \Delta(r)^{1/2} \int_G f(u) g(u^{-1}r) du h(r^{-1}st) k(r^{-1}s, t) dr \\
&= \int_G f(u) \int_G \Delta(r)^{1/2} g(u^{-1}r) h(r^{-1}st) k(r^{-1}s, t) dr du \\
&= \int_G f(u) \int_G \Delta(r)^{1/2} g(r) h(r^{-1}u^{-1}st) k(r^{-1}u^{-1}s, t) dr du \\
&= \int_G f(u) ((\Delta^{1/2}g \boxtimes h) * k)(u^{-1}s, t) du \\
&= \int_G f(u) \left( \theta(j_{\mathcal{B}}(g)j_G(h)) * k \right)(u^{-1}s, t) du \\
&= \left( \Phi(f) \left( \theta(j_{\mathcal{B}}(g)j_G(h)) * k \right) \right)(s, t). \quad \square
\end{aligned}$$

*Remark 4.3.* We can interpret the above as confirmation that Katayama duality for normal coactions is a quotient of Katayama duality for maximal ones:  $X$  can be viewed as a  $C^*(\mathcal{B}) \rtimes_{\delta} G \rtimes_{\widehat{\delta}} G - C^*(\mathcal{B})$  imprimitivity module, and  $X_R$  as a  $C_r^*(\mathcal{B}) \rtimes_{\delta^n} G \rtimes_{\widehat{\delta^n, r}} G - C_r^*(\mathcal{B})$  imprimitivity module, and then the Rieffel Surjection  $\Upsilon : X \rightarrow X_R$  of Theorem 3.2 is compatible with the regular representations  $C^*(\mathcal{B}) \rtimes_{\delta} G \rtimes_{\widehat{\delta}} G \rightarrow C_r^*(\mathcal{B}) \rtimes_{\delta^n} G \rtimes_{\widehat{\delta^n, r}} G$  and  $C^*(\mathcal{B}) \rightarrow C_r^*(\mathcal{B})$ . This follows from Theorem 4.2: we only need to observe the following equivariant isomorphisms:

$$(4.3) \quad (C^*(\mathcal{B} \times G), \alpha) \cong (C^*(\mathcal{B}) \rtimes_{\delta} G, \widehat{\delta}) \cong (C_r^*(\mathcal{B}) \rtimes_{\delta^n} G, \widehat{\delta^n}),$$

which pass to the crossed products, and hence to the reduced crossed products, and then apply Theorem 3.2.

The isomorphisms (4.3) imply the known result (see, e.g., [6, Theorem 4.1])

$$(4.4) \quad j_{\mathcal{B}}(C^*(\mathcal{B})) = (C^*(\mathcal{B}) \rtimes_{\delta} G)^{\widehat{\delta}},$$

and hence we have a commutative diagram

$$(4.5) \quad \begin{array}{ccc} & C^*(\mathcal{B}) & \\ j_{\mathcal{B}} \swarrow & & \searrow \Lambda \\ (C^*(\mathcal{B}) \rtimes_{\delta} G)^{\widehat{\delta}} & \xrightarrow[\Xi \circ \theta]{\cong} & C_r^*(\mathcal{B}) \end{array}$$

**4.2. Actions.** The following corollary appears to be new in its full generality; it is certainly well-known in the special case that  $G$  is compact. Also, the case  $A = \mathbb{C}$  (and arbitrary  $G$ ) is [9, Example 2.1]. Echterhoff and Emerson prove a special case

[1, Theorem 2.14] where  $A$  is fibered over a proper  $G$ -space. Our techniques do not require any hypotheses on the action of  $G$  on  $A$ .

**Corollary 4.4.** *If  $\beta : G \rightarrow \text{Aut } A$  is an action on a  $C^*$ -algebra, then the tensor-product action  $\beta \otimes \text{Ad } \rho$  of  $G$  on  $A \otimes \mathcal{K}(L^2(G))$  is saturated and proper in Rieffel's sense, and the generalized fixed point algebra  $(A \otimes \mathcal{K}(L^2(G)))^{\beta \otimes \text{Ad } \rho}$  is isomorphic to the reduced crossed product  $A \rtimes_{\beta, r} G$ .*

*Proof.* In diagram (4.5), we take the Fell bundle  $\mathcal{B} \rightarrow G$  to be the semidirect-product bundle

$$A \rtimes G \rightarrow G.$$

Then we have an equivariant isomorphism

$$(C^*(\mathcal{B}), \delta) \cong (A \rtimes_{\beta} G, \widehat{\beta}).$$

Thus we have a commutative diagram

$$\begin{array}{ccc} & A \rtimes_{\beta} G & \\ j_{A \rtimes_{\beta} G} \swarrow & & \searrow \Lambda \\ (A \rtimes_{\beta} G \rtimes_{\widehat{\beta}} G)^{\widehat{\beta}} & \xrightarrow[\Xi \circ \theta]{\cong} & A \rtimes_{\beta, r} G. \end{array}$$

The result now follows from the equivariant isomorphism of Imai-Takai duality:

$$(A \rtimes_{\beta} G \rtimes_{\widehat{\beta}} G, \widehat{\widehat{\beta}}) \cong (A \otimes \mathcal{K}(L^2(G)), \beta \otimes \text{Ad } \rho).$$

Note: there is a subtlety here: we have freely passed from equivariant isomorphism between proper and saturated actions to an isomorphism between the generalized fixed-point algebras; but Rieffel's generalized fixed-point algebras depend upon the choice of a suitable dense  $*$ -subalgebra. However, there is no problem in our case, because we always choose the “canonical” subalgebra associated to the obvious nondegenerate equivariant homomorphism of  $C_0(G)$  into the multiplier algebra; then the isomorphisms follow from [6, Proposition 2.6], modulo the correction in [2, Proposition 2.4].  $\square$

**4.3.  $C^*$ -bundles.** Here we specialize to the case where  $\mathcal{X} = X$  is a *space* and  $\mathcal{A}$  is just a  $C^*$ -bundle over  $X$ , so that  $C^*(\mathcal{A}) = \Gamma_0(\mathcal{A})$ . Then the orbit bundle is the  $C^*$ -bundle  $\mathcal{B} \rightarrow Y$ , where  $Y = G \backslash X$ .

**Proposition 4.5.** *If a group  $G$  acts freely and properly on a  $C^*$ -bundle  $\mathcal{A} \rightarrow X$  over a space  $X$ , then the surjection*

$$\Phi : C^*(G \backslash \mathcal{A}) \rightarrow C^*(\mathcal{A})^{\alpha}$$

*from Theorem 3.1 is an isomorphism.*

*Proof.* With the notation used in Theorem 3.2, the groupoid  $\mathcal{Y} = G \backslash \mathcal{X}$  coincides with its unit space  $Y = \mathcal{Y}^0 = \mathcal{Y}$ , which of course acts trivially on the space  $X = \mathcal{X}^0 = \mathcal{X}$ , consequently the transformation groupoid  $\mathcal{Y} * \mathcal{X}^0$  can be identified with  $X$ . The transformation bundle  $\mathcal{A} = \mathcal{B} * \mathcal{X}^0$  can be identified with  $\mathcal{B} * X$ , and every section  $a \in \Gamma_c(\mathcal{B} * X)$  is of the form

$$a(x) = (a_1(q(x)), x),$$

where  $a_1 \in \Gamma_c(\mathcal{B})$ . For  $f \in \Gamma_c(\mathcal{B})$  and  $a \in \Gamma_c(\mathcal{B} * X)$  we have

$$(\Phi(f)a)_1(x) = f(q(x))a_1(x),$$

so

$$\begin{aligned} (\Phi(f)a)(x) &= (f(q(x))a_1(x), x) \\ &= (f(q(x)), x)(a_1(x), x) \\ &= q^*(f)(x)a(x), \end{aligned}$$

where we define  $q^*(f) \in \Gamma_b(\mathcal{B} * X)$

$$(\Phi(f)a)(x) = (f(q(x)), x).$$

Thus  $\Phi(f)$  acts on  $\Gamma_c(\mathcal{B} * X)$  by pointwise multiplication by the continuous bounded section  $\Phi(f) \in \Gamma_b(\mathcal{B} * X)$  given by

$$\Phi(f)(x) = (f(q(x)), x).$$

Since  $\Gamma_b(\mathcal{B} * X)$  embeds isometrically into the multiplier algebra  $M(\Gamma_0(\mathcal{B} * X))$ , it follows that  $\Phi : \Gamma_c(\mathcal{B}) \rightarrow M(\Gamma_0(\mathcal{B} * X))$  is isometric, and hence the extension to  $\Gamma_0(\mathcal{B}) = C^*(\mathcal{B})$  is an isomorphism onto its image  $C^*(\mathcal{B} * X)^\alpha$ .  $\square$

Proposition 4.5 and Theorem 3.1 immediately imply the following corollary, which is surely folklore, although we could not find it in the literature:

**Corollary 4.6.** *If a group  $G$  acts freely and properly on a  $C^*$ -bundle  $\mathcal{A} \rightarrow X$  over a space  $X$ , then the regular representation*

$$\Lambda : \Gamma_0(\mathcal{A}) \rtimes_\alpha G \rightarrow \Gamma_0(\mathcal{A}) \rtimes_{\alpha, r} G$$

*is an isomorphism.*

Theorem 4.5 and Corollary 4.6 allow us to recover [8, Theorem 2.2]:

**Corollary 4.7.** *If a group  $G$  acts freely and properly on a locally compact Hausdorff space  $X$  and also on a  $C^*$ -algebra  $A$ , then the crossed product  $C_0(X, A) \rtimes G$  is Morita equivalent to the generalized fixed point algebra  $C_0(X, A)^\alpha$ .*

*Proof.* This follows by applying the above results to the trivial  $C^*$ -bundle  $A \times X \rightarrow X$ , since  $\Gamma_0(A \times X) \cong C_0(X, A)$ .  $\square$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287

*E-mail address:* `kaliszewski@asu.edu`

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IA 52242

*E-mail address:* `pmuhly@math.uiowa.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287

*E-mail address:* `quigg@asu.edu`

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755

*E-mail address:* `dana.williams@dartmouth.edu`